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GENUS TWO EXTREMAL SURFACES: EXTREMAL DISCS, ISOMETRIES AND WEIERSTRASS POINTS

BY

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ABSTRACT

It is known that the largest disc that a compact hyperbolic surface of genus g may contain has radius $R = \cosh^{-1}(1/2\sin(\pi/(12g-6)))$. It is also known that the number of such (extremal) surfaces, although finite, grows exponentially with g. Elsewhere the authors have shown that for genus $g > 3$ extremal surfaces contain only one extremal disc.

Here we describe in full detail the situation in genus 2. Following results that go back to Fricke and Klein we first show that there are exactly nine different extremal surfaces. Then we proceed to locate the various extremal discs that each of these surfaces possesses as well as their set of Weierstrass points and group of isometries.

1. Introduction and statement of results

1.1 EXTREMAL DISCS. It is natural to expect that discs of maximum radius in compact hyperbolic surfaces of given genus g must occur in those surfaces which admit as Dirichlet domain a regular N-gon with the largest possible number of sides $N = 12g - 6$. This result has been actually proved by M. Näätänen ([13]) and C. Bavard ([2]). It turns out that the radius R_g of such **extremal discs** is given by

$$\cosh R_g = \frac{1}{2\sin\frac{\pi}{12g-6}}.$$

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It follows that all surfaces of a given genus containing extremal discs, let us call them **extremal surfaces**, are obtained out of the regular N-gon via any coherent side-pairing identification. As a matter of fact the number of the relevant side pairings, for general g, has been studied by several authors. To our knowledge, it first appears in the article of Macbeath ([11]). It also occurs in the fundamental work of Harer and Zagier ([8]) and more recently this number has been studied by R. Bacher and A. Vdovina ([1]) in their study of one vertex triangulations of oriented surfaces. The explicit expression they obtain for the number of non-equivalent such side pairings grows very fast: it is 9 for $g = 2$, but it has 5, 7, 10 digits for genera 3, 4, 5 respectively.

Elsewhere [6] we have shown that extremal surfaces possess a unique extremal disc if $g > 3$, but that may contain several if $g = 2, 3$.

In this article we describe in detail the extremal discs, the isometries and the Weierstrass points of the nine extremal surfaces of genus 2.

1.2 THE FRICKE–KLEIN CONFIGURATIONS OF THE 18-GON. The extremal surfaces of genus 2, or equivalently the admissible side-pairings of the 18-gon, were discovered by Fricke and Klein about a century ago (see also [9], [14]). Indeed in their book [5] they found all possible ways of identifying the sides of the 18-gon. We reproduce them in Figure 1 as they are found on page 267 of [5], except for their hyperbolic, instead of circular, shape. Identification of two sides is indicated by a line connecting them.

On page 266 (second paragraph) of [5] the reader is warned that configurations obtained from the previous ones by mirror image are regarded as equivalent. Since in this paper isomorphism is going to mean orientation preserving isometry, we must consider these side pairings too. We shall show that by doing it we will add only one more to the list (in agreement with the number nine given in [1]). We shall denote this ninth side-pairing by P_8^-, since it is obtained from P_8 by reflection across the geodesic joining the middle points of sides 1 and 10 (see Figure 1).

From now on we shall denote by P_i the polygon P endowed with the i-th side-pairing, and by X_i the surface P_i uniquely determines. By what has been said, X_i is an extremal surface with an extremal disc centered at the origin o. This disc has radius R_2, R from now on, whose value is approximately $R = 1.71911$.

On each of the polygons P_i (see Figure 1) we have marked points of two different sizes. The fat ones will account for the centers of the various extremal discs in the surface X_i whereas the thin ones will correspond to the Weierstrass points. The precise location of the centers denoted q is determined by its distance to the origin $d(o, q)$, which is approximately 0.777654 in P_4, 1.28738 in P_5, and 1.21231 in P_7.

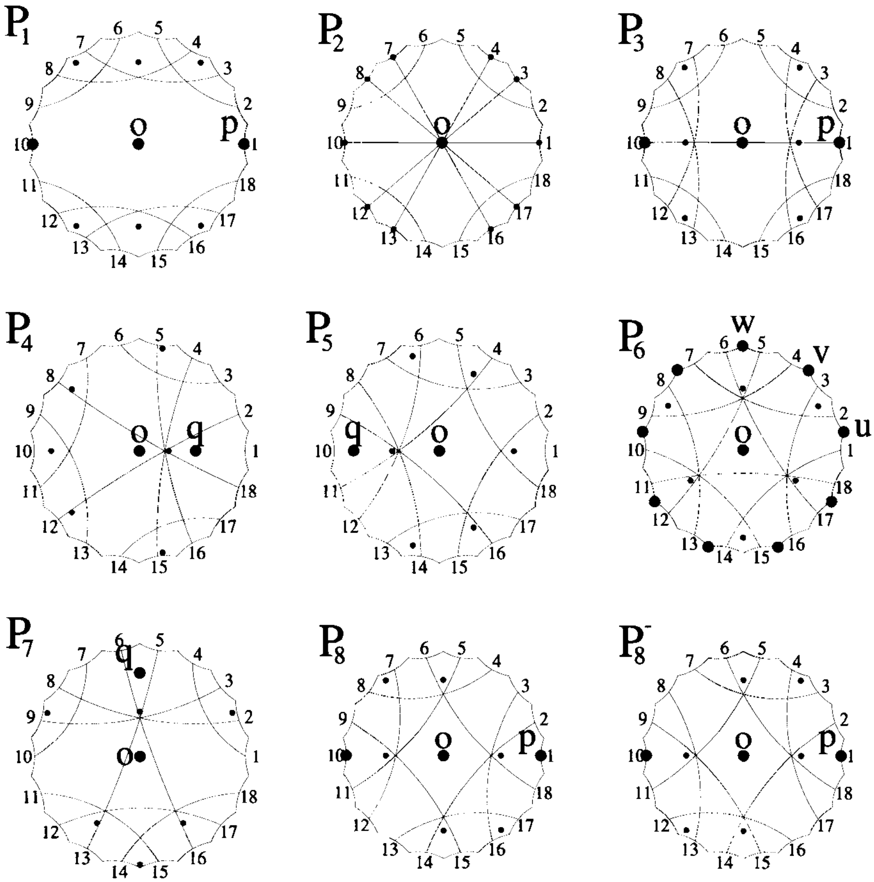

Figure 1. Extremal surfaces of genus 2: centers of extremal discs (•), and Weierstrass points (.).

1.3 Statement of results. The way the problem is approached here, knowledge of the group of isometries is both a tool and a consequence of our understanding of the extremal discs. We recall that every compact surface of genus 2 admits a unique automorphism of order 2 fixing 6 points, the **hyperelliptic involution**, which we shall denote by J.

For instance, direct inspection shows that the rotation through angle π at o, R_π, induces the hyperelliptic involution J on X_2 (fixes o and the middle points of the five pairs of sides which are identified to their opposite ones), an involution σ_2 different from J on X_1, X_3 and X_8 (fixes only o and p), and no automorphism on the remaining surfaces (R_π is not compatible with the side pairing). Similarly, $R_{2\pi/3}$ induces an automorphism σ_3 only in X_6.

THEOREM 1:

(1) *Up to isomorphism, the extremal surfaces in genus 2 are precisely* $X_1, \ldots, X_8, X_8^-$.

(2) *The list of extremal discs and group of automorphisms,* Aut, *of each of these surfaces is collected in the following table:*

Surface	*Number of discs*	*Centers*	Aut	*Generators for* Aut
X_1	2	o, p	$\mathbb{Z}_2 \times \mathbb{Z}_2$	J, σ_2
X_2	1	o	$\mathbb{Z}_2$	J
X_3	2	o, p	$\mathbb{Z}_2 \times \mathbb{Z}_2$	J, σ_2
X_4	2	o, q	$\mathbb{Z}_2$	J
X_5	2	o, q	$\mathbb{Z}_2$	J
X_6	4	o, u, v, w	D_6	$\tau, J\sigma_3$
X_7	2	o, q	$\mathbb{Z}_2$	J
X_8	2	o, p	$\mathbb{Z}_2 \times \mathbb{Z}_2$	J, σ_2
X_8^-	2	o, p	$\mathbb{Z}_2 \times \mathbb{Z}_2$	J, σ_2

where the notation for the centers is that employed in Figure 1 and τ is a certain automorphism of order 2 different from J

(3) *In each case the group of automorphisms acts transitively on the set of extremal discs.*

There are two key ideas in the proof of these results.

(a) To discover hidden discs in a given surface, we first look for automorphisms. Then their action on the explicit extremal disc centered at o will uncover new extremal discs (except, of course, for the automorphisms fixing o).

(b) To show that the discs so found exhaust the list of extremal discs we use the fact (already employed by C. Bavard in [2]) that being the center of an extremal disc imposes on a point $z \in \mathbb{D}$ certain restrictions on the amount this point is displaced (**displacement**) under the group K of isometries of $\mathbb{D}$ generated by the side pairing transformations.

(For instance, $d(z, \gamma(z))$ must be $\geq 2R$, since z and $\gamma(z)$ serve as centers of two non-overlapping discs of radius R inside $\mathbb{D}$; see Lemma 3 below.)

Accordingly the paper is organized as follows.

In section 2 we deal with point (a) and prove several statements about the automorphisms of these surfaces which will turn out to be very useful.

In section 3 we address question (b) and so we collect several results in hyperbolic trigonometry that will be used in the proof of our main theorem. The proof itself will be carried out in section 4.

In section 5 we deal with the **Weierstrass points** (fixed points of the hyperelliptic involution J) of the extremal surfaces. Since J permutes extremal discs, hence their centers, and by now we shall have found all such, we will be at this point in a position to work out the explicit formula for J, or rather a lift of it, $\widetilde{J}$. This, in turn, will allow us to detect the Weierstrass points, which are represented as the small points marked in Figure 1 (see Proposition 1).

2. Preliminary results on automorphisms

Let us denote by R_θ the rotation through angle θ at the center of the polygon.

We shall prove the following result relative to the group of automorphisms, $\mathrm{Aut}(X_i)$, of the surfaces under consideration.

LEMMA 1:

(i) *The rotation through angle π, R_π, induces an automorphism, σ_2, only for the surfaces X_1, X_2, X_3, X_8 and X_8^-. This automorphism agrees with the hyperelliptic involution J only for X_2.*

(ii) *For $\theta = 2\pi/3$, R_θ induces an automorphism only for the surface X_6. This automorphism σ_3 has order 3 and fixes the four points o, u, v, w.*

This surface admits an automorphism of order 2, τ, different from J. The group generated by τ and J acts transitively on the set $\{o, u, v, w\}$.

(iii) *No other rotation R_θ, $\theta \neq \pi, 2\pi/3$, induces an automorphism of any of the surfaces $X_1, \ldots, X_8, X_8^-$.*

Proof: (i) Direct inspection shows that for surfaces $X_1, X_2, X_3, X_8, X_8^-$ the rotation R_π is compatible with the side pairing, while for the remaining ones it is not (e.g., in P_7 the middle points of sides 5 and 13 are paired while their images under R_π, the middle points of sides 14 and 4, are not).

The automorphism induced by R_π on surfaces $X_1, X_2, X_3, X_8, X_8^-$ agrees with the hyperelliptic involution when it fixes exactly six points, the Weierstrass points. A careful look at the figures shows that this only occurs in case of X_2, the set of Weierstrass points being o and the middle points of sides $1, 3, 4, 7$ and 8.

(ii) Again by direct checking we see that for $\theta = 2\pi/3$, R_θ induces an isomorphism σ_3 on surface X_6. This isomorphism has four fixed points, namely o, u, v, w.

Now from the classification of Riemann surfaces of genus 2 according to its automorphism group, going back to Bolza ([4], see also [15] and [10]), we see that X_6 must be the Riemann surface of an algebraic equation of the form $y^2 = (x^3 - 1)(x^3 - \lambda^6)$, for some λ with $\lambda^6 \neq 1$, and that σ_3 corresponds to the automorphism $\sigma_3(x, y) = (\xi_3 x, y)$, where ξ_3 is a primitive 3-root of unity, the hyperelliptic involution being given by the expression $J(x, y) = (x, -y)$.

The four points of the algebraic curve fixed by σ_3, to match o, u, v, w are $(0, \lambda^3), (0, -\lambda^3)$ and the two points at infinity, ∞_1, ∞_2 that the Riemann surface of this curve has (see [12], 3.13).

We now observe that this algebraic curve admits yet another automorphism of order 2 different from J, namely $\tau(x, y) = (\lambda^2/x, \lambda^3 y/x^3)$.

As for the last point, we recall that J permutes the two point sets $\{\infty_1, \infty_2\}$ and $\{(0, \lambda^3), (0, -\lambda^3)\}$, whereas τ permutes these two sets. The proof is now complete.

(iii) The last statement follows by direct inspection on Figure 1, since no rotation different from the ones related above is compatible with the side-pairing identifications. ∎

LEMMA 2:

(i) *Let $f\colon X_i \longrightarrow X_j$ be an isomorphism between two of the nine surfaces in question, and suppose that f sends $o \in X_i$ to $o \in X_j$. Then, f is realized as a rotation of the polygon.*

In particular, if the point o is a Weierstrass point, then the hyperelliptic involution $J\colon X \longrightarrow X$ is given by the rotation R_π.

(ii) *Except for the polygon P_2, P_i represents an extremal surface possessing, at least, two different discs centered at o and $J(o)$.*

Proof: (i) Let us denote by K_i, K_j the groups generated by the side pairing transformations defining X_i and X_j respectively. A lift $\tilde{f}\colon \mathbb{D} \longrightarrow \mathbb{D}$ of f can be chosen so that $\tilde{f}(0) = 0$, i.e., such that $\tilde{f} = R_\theta$ for some angle θ. We have to show that R_θ preserves the polygon P, which means that θ is an integer multiple of $2\pi/18$.

This can be seen by observing that if p, p' are the middle points of two sides which get paired under the action of K_i, their images $R_\theta(p), R_\theta(p')$ must get paired under the action of K_j. Since both points lie inside the inscribed disc, this is only possible if $R_\theta(p), R_\theta(p')$ are the middle points of certain sides of the polygon.

(ii) By Lemma 1 (i) and the previous statement, the center of P_i for $i \neq 2$ is not a Weierstrass point. Hence J maps the extremal disc centered at o into a different one. ∎

3. Hyperbolic trigonometry

In this section we use [3] as a general reference for hyperbolic trigonometry.

LEMMA 3: *Let $\{\gamma_{(i,j)}\}$ be the set of side-pairings on the regular polygon P with 18 sides defining an extremal Riemann surface of genus 2. Let L be the length of the sides of P, and H the distance from the center of P to any of the vertices.*

If $z \in P$ represents the center of an extremal disc embedded into the surface, then the displacement of z by every $\gamma_{(i,j)}$ verifies $d(z, \gamma_{(i,j)}(z)) \geq 2R$. If for some $\gamma_{(i_0,j_0)}$ the inequality is strict, then $d(z, \gamma_{(i_0,j_0)}(z)) \geq 2H + L$.

Proof: Let $\tilde{P}_0$ be the Dirichlet domain centered at z for the Fuchsian group G generated by the transformations $\{\gamma_{(i,j)}\}$. Since z is the center of an extremal disc, $\tilde{T} = \{g(\tilde{P}_0)\}_{g\in G}$ yields a tessellation of $\mathbb{D}$ by regular hyperbolic polygons with $12g-6$ sides. The image of z under $\gamma_{(i,j)}$ is the center of some polygon $\tilde{P}_1$ of the tessellation $\tilde{T}$.

It is clear that $d(z, \gamma_{(i,j)}(z))$ attains its minimum $2R$ if and only if $\tilde{P}_0$ and $\tilde{P}_1$ are adjacent. If this is not the case, then $d(z, \gamma_{(i,j)}(z))$ equals at least $2H + L$ (see Figure 2). ∎

In what follows, we will denote

$$d_1 = 2H + L. \tag{1}$$

It is not difficult to show that the numerical value of d_1 equals approximately 4.74604.

We will need some relations between these magnitudes. For instance, suitable application of the hyperbolic cosine rule to the triangles shown in Figure 2 give

$$\cosh 2R = \cosh^2 H - \sinh^2 H \cos(2\pi/3), \tag{2}$$

$$\cosh 2R = \cosh H \cosh(H+L) - \sinh H \sinh(H+L)\cos(\pi/3), \tag{3}$$

$$\cosh 2R = \cosh R \cosh(d_1/2). \tag{4}$$

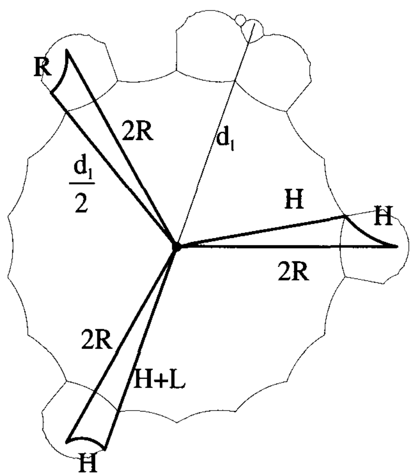

Figure 2. Some relevant distances.

The next lemma is related to the convexity of the hyperbolic metric in $\mathbb{D}$:

LEMMA 4: *Let R be a hyperbolic triangle with vertices at z_1, z_2, z_3, and g a hyperbolic isometry. Then*

$$d(z, gz) \leq \max\{d(z_1, gz_1), d(z_2, gz_2), d(z_3, gz_3)\}$$

for all $z \in R$.

Proof: Denote the axis of g by γ_g, and its translation length by T_g. Then, by the well known formula $\sinh \frac{1}{2}d(z, gz) = \cosh d(z, \gamma_g) \sinh \frac{1}{2}T_g$ (see [3], page 174), it is enough to show $d(z, \gamma_g) \leq \max\{d(z_1, \gamma_g), d(z_2, \gamma_g), d(z_3, \gamma_g)\}$.

This inequality holds trivially if $z = z_1$. Now, for z in $[z_2, z_3]$, we have

$$d(z, \gamma_g) < \max\{d(z_2, \gamma_g), d(z_3, \gamma_g)\}$$

(this is because the set $\{z \in \mathbb{D} \text{ s.t. } d(z, \gamma_g) \leq k\}$ is convex in the hyperbolic metric, see [3], pages 163, 139). The result follows applying the same argument to any point in the segment $[z_1, w]$, where $w \in [z_2, z_3]$. ∎

LEMMA 5: *Let us label the sides of the regular polygon P with numbers from 1 to 18. Let p_i be the middle point of side labeled i, z_i the vertex between sides i and $i+1$, and let $\gamma_{(i,i+n)}$ be the orientation preserving isometry of $\mathbb{D}$ sending side* *labeled i to side labeled $i+n$ ($1 \le n \le 17$), where notation should be understood modulo 18. Then*

(i) $d(p_i, \gamma_{(i,i+n)}(p_i)) \le 2R$, *and the inequality is strict unless* $n = 9$.

(ii) $d(z_i, \gamma_{(i,i+n)}(z_i)) < 2R$ *if* $n < 7$ *or* $n > 13$, $d(z_i, \gamma_{(i,i+n)}(z_i)) = 2R$ *if* $n = 7, 13$, *and* $d(z_i, \gamma_{(i,i+n)}(z_i)) > 2R$ *in the other cases.*

(iii) $d(p_{i+1}, \gamma_{(i,i+n)}(p_{i+1})) < 2R$ *if* $n < 6$ *or* $n > 15$, $d(p_{i+1}, \gamma_{(i,i+n)}(p_{i+1})) = 2R$ *if* $n = 6, 15$, *and* $d(p_{i+1}, \gamma_{(i,i+n)}(p_{i+1})) > 2R$ *in the other cases.*

(iv) $d(z_{i+1}, \gamma_{(i,i+n)}(z_{i+1})) < 2R$ *if* $n < 5$, $d(z_{i+1}, \gamma_{(i,i+n)}(z_{i+1})) = 2R$ *if* $n = 5, 17$, *and* $d(z_{i+1}, \gamma_{(i,i+n)}(z_{i+1})) > 2R$ *in the other cases.*

(v) $d(z_{i-1}, \gamma_{(i,i+n)}(z_{i-1})) < 2R$ *if* $n < 5$ *or* $n > 11$, $d(z_{i-1}, \gamma_{(i,i+n)}(z_{i-1})) = 2R$ *if* $n = 5, 11$, *and* $d(z_{i-1}, \gamma_{(i,i+n)}(z_{i-1})) > 2R$ *in the other cases.*

(vi) $d(p_{i-1}, \gamma_{(i,i+n)}(p_{i-1})) < 2R$ *if* $n < 3$ *or* $n > 12$, $d(p_{i-1}, \gamma_{(i,i+n)}(p_{i-1})) = 2R$ *if* $n = 3, 12$, *and* $d(p_{i-1}, \gamma_{(i,i+n)}(p_{i-1})) > 2R$ *in the other cases.*

(vii) $d(z_{i-2}, \gamma_{(i,i+n)}(z_{i-2})) < 2R$ *if* $n > 13$ *or* $n > 11$, $d(z_{i-2}, \gamma_{(i,i+n)}(z_{i-2})) = 2R$ *if* $n = 1, 13$, *and* $d(z_{i-2}, \gamma_{(i,i+n)}(z_{i-2})) > 2R$ *in the other cases.*

In addition, the displacement of $p_{i-1}, z_{i-1}, p_i, z_i$ *and* p_{i+1} *under* $\gamma_{(i,i+n)}$ *is less than* d_1 *for all* n, $d(z_{i-2}, \gamma_{(i,i+n)}(z_{i-2})) \le d_1$ *with strict inequality unless* $n = 7$, *and* $d(z_{i+1}, \gamma_{(i,i+n)}(z_{i+1})) \le d_1$ *with strict inequality unless* $n = 11$.

Proof: We apply successively the hyperbolic cosine rule to the triangle with vertices at o, x and $\gamma_{(i,i+n)}(x)$, where x is the relevant point in each case (p_i for (i), z_i for (ii), etc. ...). We obtain:

(i) Is just a consequence of the triangular inequality.

(ii) $\cosh d(z_i, \gamma_{(i,i+n)}(z_i)) = \cosh^2 H - \sinh^2 H \cos(2(n-1)\pi/18)$, so comparison with equation 2 gives the result.

(iii) We have

$$\cosh d(p_{i+1}, \gamma_{(i,i+n)}(p_{i+1})) = \cosh R \cosh(d_1/2) - \sinh R \sinh(d_1/2) \cos((2n-3)\pi/18).$$

The statement follows from equation (4).

(iv) As in the previous cases we can compute

$$\cosh d(z_{i+1}, \gamma_{(i,i+n)}(z_{i+1})) = \cosh H \cosh(H+L) - \sinh H \sinh(H+L) \cos((2n-4)\pi/18).$$

Comparison with equation (3) proves the result.

The proof of (v), (vi), and (vii) follows clearly from (ii), (iii) and (iv) by applying a reflection on the polygon P, and the last statement follows trivially from the triangular inequality (keeping in mind equation (1)). ∎

This lemma allows us to state the hyperbolic trigonometry computations in the form we will use them in the proof of the main theorem.

Denoting by $P(x_1, \dots, x_k)$ the hyperbolic closed polygon whose vertices are $x_1, \dots, x_k$, we have proved the following facts (remember $d(o, \gamma_{(i,j)}(o)) = 2R$ for every i, j):

(F3) The displacement of each point in $P(p_{i-1}, z_{i-1}, z_i, z_{i+1}, o)$ under $\gamma_{(i,i+3)}$ is less than $2R$, except for p_{i-1} and o, which are displaced exactly $2R$.

(F4) Every point in $P(z_{i-1}, z_i, z_{i+1}, o) \setminus \{o\}$ is displaced strictly less than $2R$ under $\gamma_{(i,i+4)}$.

(F5) The displacement of o, z_{i-1}, z_{i+1} under $\gamma_{(i,i+5)}$ equals $2R$. All other points in $P(z_{i-1}, z_i, z_{i+1}, o)$ are displaced less than $2R$ under $\gamma_{(i,i+5)}$.

(F6) Every $z \in P(p_i, z_i, p_{i+1}, o)$ is displaced less than $2R$ under $\gamma_{(i,i+6)}$, except o and p_{i+1} which are displaced $2R$.

(F7) z_i and o are displaced $2R$ under $\gamma_{(i,i+7)}$ while any other point in $P(p_i, z_i, o)$ is displaced less than $2R$.

The next five facts follow from the previous ones by reflection:

(F15) The displacement of each point in $P(z_{i-2}, z_{i-1}, z_i, p_{i+1}, o)$ by $\gamma_{(i,i+15)}$ is less than $2R$, except for p_{i+1} and o, which are displaced exactly $2R$.

(F14) Every point in $P(z_{i-2}, z_{i-1}, z_i, o) \setminus \{o\}$ is displaced strictly less than $2R$ under $\gamma_{(i,i+14)}$.

(F13) The displacement of o, z_{i-2}, z_i under $\gamma_{(i,i+13)}$ equals $2R$. All other points in $P(z_{i-2}, z_{i-1}, z_i, o)$ are displaced less than $2R$ under $\gamma_{(i,i+13)}$.

(F12) Every $z \in P(p_{i-1}, z_{i-1}, p_i, o)$ is displaced less than $2R$ under $\gamma_{(i,i+12)}$, except o and p_{i-1} which are displaced $2R$.

(F11) z_{i-1} and o are displaced $2R$ under $\gamma_{(i,i+11)}$ while any other point in $P(z_{i-1}, p_i, o)$ is displaced less than $2R$.

We will need two more facts:

(F9) For every $z \in P(z_{i-2}, z_{i-1}, z_i, z_{i+1}, o)$ we have $d(z, \gamma_{(i,i+9)}(z)) \geq 2R$, with equality if and only if z lies in the segment $[o, p_i]$.

(F*) For every $z \in P(z_{i-2}, z_{i-1}, z_i, z_{i+1}, o)$ we have $d(z, \gamma_{(i,i+n)}(z)) \leq d_1$, with equality if and only if either $n = 7$, $z = z_{i-2}$ or $n = 11$, $z = z_{i+1}$.

4. Proof of Theorem 1

We now proceed to prove each of the three statements contained in Theorem 1, beginning with the second one:

Part (2) We examine one by one the nine surfaces. In our search for centers of extremal discs, suitable combination of Lemma 3 with the F properties developed in the previous section will enable us to discard entire polygonal regions. We will use the term **extra** or **hidden** extremal disc to refer to an extremal disc different from the self-evident one, centered at o.

As for the statement concerning the automorphism group of these surfaces, it will be in all cases a consequence of Lemma 1 and the transitivity of the automorphisms already found on the set of extremal discs.

- For $\mathbf{P_1}$ it is sufficient to apply (F3) to $\gamma_{(2,5)}$, (F15) to $\gamma_{(5,2)}$, and note the symmetry on the identification pattern. This shows that the only $z \in P$ that could be the center of a hidden extremal disc is p_1 (equivalently p_{10}). Then, Lemma 2(ii) shows that X_1 has exactly two discs centered at o and $J(o) = p$.

- $\mathbf{P_2}$: Bavard ([2]) proved that X_2 contains just one extremal disc.

- $\mathbf{P_3}$: Apply (F3) to $\gamma_{(2,5)}$, (F15) to $\gamma_{(5,2)}$, and note the symmetry on the side-pairing. This shows that p_1 is the only point which could be the center of an extra extremal disc. Lemma 2(ii) yields again the result.

- $\mathbf{P_4}$: Application of (F4) to $\gamma_{(9,13)}$ and $\gamma_{(7,11)}$, (F15) to $\gamma_{(6,3)}$, (F3) to $\gamma_{(3,6)}$, and (F9) (with (F*) and Lemma 3) to $\gamma_{(1,10)}$, followed by the symmetry on the side pairings across the geodesic joining p_1 and p_{10}, show that the only points that could be centers of extremal discs must lie in the geodesic segment $[o, p_1]$.

The displacement of each $z \in [o, p_1]$ under $\gamma_{(18,8)}$ is less than d_1 by (F*). So, if a center different from o exists (and it has to exist by Lemma 2(ii)), it has to be displaced exactly $2R$ under $\gamma_{(18,8)}$.

Note that only two points in $[o, p_1]$ have this displacement: one of them is o; let us denote the other one by q. Using hyperbolic trigonometry in the quadrilateral $P(o, q, \gamma_{(18,8)}(o), \gamma_{(18,8)}(q))$, it is not difficult to show that $d(o, q) \simeq 0.777654$.

- $\mathbf{P_5}$: Application of (F13) to $\gamma_{(2,15)}$ and $\gamma_{(5,18)}$, (F4) to $\gamma_{(3,7)}$, (F14) to $\gamma_{(7,3)}$, (F6) to $\gamma_{(8,14)}$, and (F9) (with (F*) and Lemma 3) to $\gamma_{(10,1)}$, while keeping in mind the symmetry that the side-pairing possesses, shows that a hidden center (existing by Lemma 2(ii)) has to belong to the segment $[o, p_{10}]$ or to the triangle $P(p_8, z_7, o)$.

Now, note that by (F*), $\gamma_{(9,16)}$ displaces each point of $P(p_9, z_8, z_7, o)$ less than d_1, with the exception of z_7, which is displaced d_1. But z_7 cannot be the center of an extremal disc (apply $\gamma_{(7,3)}$), so we have only to look at the set C of points being displaced $2R$ by $\gamma_{(9,16)}$. It is known ([3], p. 163) that C is the union of two arcs of circumference C_1, C_2, one at each side of L, the axis of $\gamma_{(9,16)}$.

We shall show that L is the common perpendicular to the (complete) geodesics containing sides 9 and 16: this common perpendicular does exist, i.e., the two geodesics do not intersect (not even in the boundary of $\mathbb{D}$), for if they did at a point z, the Gauss–Bonnet formula would imply that the area of the polygon $P(z_9, z_{10}, \ldots, z_{15}, z)$ is negative. On the other hand, it is not difficult to check that the common perpendicular is invariant under the transformation $\gamma_{(9,16)}$, so it is the axis.

It is clear that L leaves the whole set $P(p_9, z_8, z_7, o)$ at the same side (in particular, it lies above L); otherwise the area of the 4-gon defined by L, $[p_9, p_{16}]$ and the geodesics containing sides 9 and 16 would be negative. Therefore only one of the two arcs C_1 and C_2 passes through $P(p_8, z_8, z_7, o)$. Indeed, by symmetry, L is orthogonal to the geodesic containing $[z_3, z_{12}]$. This shows that the set of points in $P(p_9, z_8, z_7, o)$ displaced $2R$ by $\gamma_{(9,16)}$ is an arc of circumference passing through o and tangent to the segment $[p_8, p_{17}]$. Therefore no point in $P(p_8, z_7, o)$ can be the center of a hidden extremal disc.

So, we have only to check $[o, p_{10}]$. Notice that (F*) shows that any point in this segment is displaced less than d_1 under $\gamma_{(9,16)}$, and there are precisely two points being displaced exactly $2R$, namely o and, say, q. In fact, it is not difficult to find $d(o,q)$ using hyperbolic trigonometry in the quadrilateral $P(o, q, \gamma_{(9,16)}(o), \gamma_{(9,16)}(q))$ as in case of P_4, resulting in approximately 1.28738.

We finish noting that q has to be the center of an extremal disc by Lemma 2(ii).

- $\mathbf{P_6}$: Note that the side-pairing has an order 3 symmetry. So, it is only needed to look at the subset of P_6 defined by $P(z_1, z_2, \ldots, z_7, o)$.

Apply (F13) to $\gamma_{(7,2)}$ and $\gamma_{(3,16)}$, and (F5) to $\gamma_{(4,9)}$. We see that the only points that could be the center of an extremal disc are z_1, z_3, and z_5, which are the points marked u, v, w in P_6. Part (ii) of Lemma 1 shows that these are indeed centers of hidden extremal discs.

- $\mathbf{P_7}$: Notice that the side pairing is, as in the two former cases, symmetric (with the line joining z_5 and z_{14} as symmetry axis).

Applying (F15) to $\gamma_{(4,1)}$, (F14) to $\gamma_{(18,14)}$, $\gamma_{(15,11)}$, (F13) to $\gamma_{(17,12)}$, (F3) to $\gamma_{(1,4)}$, and the fact that the side pairings are symmetric, we see that there are no centers of hidden extremal discs in $P(p_6, z_6, \ldots, z_{18}, z_1, \ldots, z_4, p_5, o)$. So, we only need to check if there could be an extremal disc in $P(p_5, z_5, p_6, o)$.

To deal with the remaining, still symmetric, polygon $P(p_5, z_5, p_6, o)$, we proceed as follows:

(1) We show that $\gamma_{(4,1)}$, and hence its symmetric side pairing $\gamma_{(7,10)}$, displaces the points p_5, z_5, p_6, o less than d_1. This implies that the possible centers of extremal discs must lie on the intersection of the sets C and C' consisting of points displaced exactly $2R$ under $\gamma_{(4,1)}$ and $\gamma_{(7,10)}$ respectively.

Recall ([3], p. 163) that C (resp. C') is the union of two arcs of circumference C_1, C_2 (resp. C_1', C_2') one at each side of L (resp. L'), the axis of $\gamma_{(4,1)}$ (resp. $\gamma_{(7,10)}$).

(2) The axis L (resp. L') is the common perpendicular to the geodesics containing sides 4 and 1 (resp. 7 and 10).

(3) The polygon $P(p_5, z_5, p_6, o)$ lies entirely in one of the two components of $\mathbb{D} \setminus \{L\}$ (resp. $\mathbb{D} \setminus \{L'\}$). Therefore only one of the two arcs, say C_1 (resp. C_1'), meets $P(p_5, z_5, p_6, o)$.

(4) The points p_5, o, p_{18} (resp. p_6, o, p_{11}) are displaced $2R$ by $\gamma_{(4,1)}$ (resp. $\gamma_{(7,10)}$) and lie in the same component of $\mathbb{D} \setminus \{L\}$ (resp. $\mathbb{D} \setminus \{L'\}$). Thus, we may compute the circumferences C_1, C_1' and their intersection point q explicitly, resulting in the expression given in section 1. This point must be, therefore, the hidden extremal disc that exists by Lemma 2(ii).

We now fill the gaps of each step.

(1) By applying (F*) to $\gamma_{(4,1)}$ we see that p_5, z_5 are displaced less than d_1. To draw the same conclusion for the remaining point p_6, we first work on the triangle $P(o, \gamma_{(4,1)}(z_5), \gamma_{(4,1)}(p_6))$ of which two lengths $H + L, L/2$ are known (see Figure 2), to determine the angle at the vertex o and the length of the third side, $d(o, \gamma_{(4,1)}(p_6))$. This, in turn, determines the angle at o of the triangle $P(o, p_6, \gamma_{(4,1)}(p_6))$. This second triangle allows us to calculate $d(p_6, \gamma_{(4,1)}(p_6))$. A computer calculation gives $d(p_6, \gamma_{(4,1)}(p_6)) \simeq 4.5447$, which is less than the numerical value obtained for d_1 in section 3.

(2) To show this point, argue exactly as we did in case of P_5.

(3) Let w_4, w_1 be the points at which L intersects (perpendicularly) the geodesics containing sides 4 and 1. If L intersected $P(p_5, z_5, p_6, o)$, the area of the polygon $P(w_4, w_1, z_1, z_2, z_3)$ would equal $3\pi - \pi - 3(2\pi/3) = 0$.

(4) Applying (F15) to $\gamma_{(4,1)}$ we see that p_5 is displaced $2R$, and so must be p_{18}, since it is clear that both points are at the same distance of the translation axis L and in the same connected component of $\mathbb{D} \setminus \{L\}$.

- $\mathbf{P_8}$: Apply (F12) to $\gamma_{(2,14)}$, (F13) to $\gamma_{(3,16)}$, (F4) to $\gamma_{(4,8)}$, (F14) to $\gamma_{(8,4)}$, (F6) to $\gamma_{(9,15)}$, and (F9) to $\gamma_{(10,1)}$, and note the symmetry of the side-pairing under rotation through angle π fixing o. We see that a center of an extremal disc different from o has to be either p_1 or a point in $P(z_5, z_6, o)$.

Now we compute the displacement of p_6 under $\gamma_{(4,8)}$. By the cosine law we have

$$\cosh d(p_6, \gamma_{(4,8)}(p_6)) = \cosh^2 \frac{d_1}{2} - \sinh^2 \frac{d_1}{2} \cos \frac{4\pi}{18}$$

which, as computer calculation shows, is strictly less than $\cosh 2R$.

So, no point in $P(z_5, p_6, o)$ can be the center of an extra extremal disc, and the same argument applied to $\gamma_{(8,4)}$ discards also every point in $P(p_6, z_6, o)$.

As in the previous cases, Lemma 2(ii) guarantees that p_1 has to be the center of an extremal disc, and is the only one apart from o.

- $\mathbf{P_8^-}$: Of course, our statements about X_8^- follow from those about X_8 by reflection.

Part (3). The fact that the group of automorphisms acts transitively on the set of extremal discs follows immediately from the proof of (2).

Part (1). Since the eight surfaces X_i given in [9] and [5] fill the complete list of equivalence classes of extremal surfaces modulo isomorphism or anti-isomorphism, it is clear that when we consider equivalence classes of isomorphic extremal surfaces, the list is exhausted by X_i, $i = 1, \ldots, 8$ and all their mirror images.

We next observe that two surfaces obtained in this process out of the same X_i must be isomorphic, since composition of two anti-automorphisms produces an automorphism. This reduces our problem to the study of just one mirror image of each surface X_i.

Now by choosing in each P_i one's preferred diagonal to perform the mirror image, it is seen that in all cases but P_8 we obtain an anti-automorphism of the surface X_i. Thus, the only possibly new surface added to the list in this way is the mirror image of X_8, which we had denoted X_8^-.

We next address the question of whether these nine surfaces are pairwise non-isomorphic. The information displayed in the table about number of discs and automorphism groups along with the numerical explicitness of the location of the centers, hence of the distance between them, allows us to conclude that there are no isomorphisms between our surfaces except, perhaps, for X_1, X_3, X_8, and X_8^-.

Now, suppose we had an isomorphism f between two of these surfaces. Since f must preserve extremal discs, and these surfaces have only two of them, we can apply Lemma 2(i) to conclude that either f or $J \circ f$ can be realized as a rotation of the polygon; but we directly check that no rotation induces an isomorphism between them. ∎

5. Weierstrass points

In this section we describe geometrically and give explicit expressions for the hyperelliptic involution J (or rather a lift $\widetilde{J}$ of it) on each of the surfaces under consideration. This will allow us to locate representatives of the six Weierstrass points inside P. We characterize them as fixed points of $g^{-1} \circ \tilde{J}$ for the six transformations g given in each case.

We can state the following:

PROPOSITION 1: *The table below shows the explicit expression for a lift of the hyperelliptic involution for each of the surfaces X_i, and the Weierstrass points of these surfaces. Their location is approximately the one indicated in Figure 1.*

Surface	$\tilde{J}(z)$	*Weierstrass points: $z \in P$ s.t. $\tilde{J}(z) = g(z)$*
X_1	$\frac{z-r}{1-rz}$	$g = \gamma_{(14,11)}, \gamma_{(16,12)}, \gamma_{(6,9)}, \gamma_{(4,8)}, \gamma_{(1,10)} \circ \gamma_{(13,17)}, \gamma_{(1,10)} \circ \gamma_{(7,3)}$
X_2	$-z$	$g = \mathrm{Id}, \gamma_{(1,10)}, \gamma_{(12,3)}, \gamma_{(13,4)}, \gamma_{(16,7)}, \gamma_{(17,8)}$
X_3	$-\frac{z-r}{1-rz}$	$g = \mathrm{Id}, \gamma_{(4,17)}, \gamma_{(16,3)}, \gamma_{(10,1)}, \gamma_{(10,1)} \circ \gamma_{(7,12)}, \gamma_{(10,1)} \circ \gamma_{(13,8)}$
X_4	$-\frac{z-d}{1-dz}$	$g = \mathrm{Id}, \gamma_{(5,16)}, \gamma_{(8,18)}, \gamma_{(10,1)}, \gamma_{(12,2)}, \gamma_{(15,4)}$
X_5	$-\frac{z+d}{1+dz}$	$g = \mathrm{Id}, \gamma_{(6,12)}, \gamma_{(4,11)}, \gamma_{(1,10)}, \gamma_{(16,9)}, \gamma_{(14,8)}$
X_6	$-\frac{z-hi}{1+hiz}$	$g = \mathrm{Id}, \gamma_{(2,7)}, \gamma_{(17,6)}, \gamma_{(12,5)}, \gamma_{(9,4)}, \gamma_{(17,6)} \circ \gamma_{(14,1)}$
X_7	$-\frac{z-di}{1+2diz}$	$g = \mathrm{Id}, \gamma_{(2,8)}, \gamma_{(16,6)}, \gamma_{(13,5)}, \gamma_{(9,3)}, \gamma_{(16,6)} \circ \gamma_{(14,18)}$
X_8	$-\frac{z-r}{1-rz}$	$g = \mathrm{Id}, \gamma_{(14,2)}, \gamma_{(6,18)}, \gamma_{(10,1)}, \gamma_{(16,3)}, \gamma_{(10,1)} \circ \gamma_{(7,12)}$
X_8^-	$-\frac{z-r}{1-rz}$	$g = \mathrm{Id}, \gamma_{(6,18)}, \gamma_{(14,2)}, \gamma_{(10,1)}, \gamma_{(4,17)}, \gamma_{(10,1)} \circ \gamma_{(13,8)}$

We have denoted $r = \tanh \frac{R}{2}$, $h = \tanh \frac{H}{2}$, *and* $d = \tanh \frac{d(o,q)}{2}$.

Proof:

STEP 1: In order to understand J, we first note that it permutes pairs of centers of extremal discs, and so it preserves the geodesics through them.

For the case of X_i, $i = 1, 3, 8$, let us denote by γ the oriented closed geodesic represented on P_i by the (oriented) segment $[p_{10}, p_1]$. It is clear that J leaves γ setwise invariant. It follows that if we require a lift of J, $\tilde{J}$: $\mathbb{D} \longrightarrow \mathbb{D}$, to send $p_1 = r$ ($r = \tanh(R/2)$) to o, then we must have

$$\tilde{J}(z) = \pm \frac{z - r}{1 - rz},$$

depending on whether $J(o) = r = p_1$ or $J(o) = -r = p_{10}$. So, in order to fully determine J we only need to check whether $J(\gamma) = \gamma$ or $J(\gamma) = \gamma^{-1}$.

Here we invoke the result of [7], that $J(\gamma) = \gamma$ if and only if γ is a dividing curve. It is immediate to check that γ is a dividing curve in case of X_1, while it is not dividing in case of X_3, X_8 and X_8^-. Hence for X_1 we can choose a lift $\widetilde{J}$ so that $\widetilde{J}(o) = p_{10}$, yielding

$$\widetilde{J}(z) = \frac{z - r}{1 - rz}.$$

On the other hand,

$$\widetilde{J}(z) = -\frac{z - r}{1 - rz}$$

is a lift of the hyperelliptic involution for X_3 and X_8.

In case of X_4, X_5 and X_7 there is no ambiguity, since a lift $\widetilde{J}$ sending q to o must send o to q.

For X_6 we shall prove that $J(o) = w$. Recall that P_6 is symmetric with respect to the line $[z_5, z_{14}]$. Let us call S the anti-automorphism on X_6 defined by this symmetry. Note that $S(v) = u$, while o and w remain fixed by S. The fact that J commutes with S (and indeed with any automorphism or anti-automorphism) shows that J fixes $[z_5, z_{14}]$. This confirms that $J(o) = w$, and this again determines $\widetilde{J}$ up to orientation.

Now, denote by γ the oriented closed geodesic given by the segment $[o, z_5]$ followed by side 11 and the segment $[z_{14}, o]$. Since γ is not a dividing curve, the mentioned result of [7] shows that $J(\gamma) = \gamma^{-1}$. The lift $\widetilde{J}$ is now completely determined: it sends back z_5 to o.

STEP 2: We shall find the Weierstrass points of each surface. Note that $z \in P$ is a fixed point of J if there exists an element g in the group G generated by the side pairing transformations such that $\widetilde{J}(z) = g(z)$. Thus, in order to determine the Weierstrass points of, say, X_1 we proceed as follows:

We can write explicitly the transformations $\gamma_{(j,k)}$; it is not difficult to show that

$$\gamma_{(1,10)}(z) = \frac{(1 + r^2)z - 2r}{(1 + r^2) - 2rz},$$

and any other $\gamma_{(j,k)}$ is related to $\gamma_{(1,10)}$ by means of the relation

$$\gamma_{(j,k)} = -R_{2(k-1)\pi/18} \circ \gamma_{(1,10)} \circ R_{-2(j-1)\pi/18}.$$

Now we look for solutions in P of the equation $\tilde{J}(z) = g(z)$, first when g runs among the set of side pairing transformations and then among the set of compositions of pairs of them. In this way, we find such solutions for $g = \gamma_{(14,11)}, \gamma_{(16,12)}, \gamma_{(6,9)}, \gamma_{(4,8)}, \gamma_{(1,10)} \circ \gamma_{(13,17)}$ and $\gamma_{(1,10)} \circ \gamma_{(7,3)}$. This gives the six Weierstrass points.

These computations (for each surface) may be done explicitly. For instance, the Weierstrass point in X_1 obtained as the solution of $\tilde{J}(z) = \gamma_{(16,12)}(z)$ turns out to be

$$z = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

where, if we denote by ξ the first 18-root of unity, a, b and c are given by

$$a = 2r\xi^3 + r(1 + r^2)\xi^{-4},$$
$$b = -2r^2(\xi^{-7} + \xi^3) - (1 + r^2)(1 + \xi^{-4}),$$
$$c = 2r\xi^{-7} + r(1 + r^2).$$

This way we are able to locate them as shown in Figure 1. ∎

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